

# Resonances of the Quantum $\delta$ -Kicked Accelerator

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(Dated: February 1, 2008)

We report the observation of high order resonances of the quantum  $\delta$ -kicked accelerator using a BEC kicked by a standing wave of light. The signature of these resonances is the existence of quantum accelerator modes. For the first time quantum accelerator modes were seen near  $1/4$  and  $1/3$  of the Talbot time. Using a BEC enabled us to study the detailed structure of the modes and resonances which are related to the fractional Talbot effect. We present a general theory for this system and apply it to predict the behavior of the accelerator modes.

PACS numbers: 05.45.Mt, 03.75.Lm, 32.80.Lg, 42.50.Vk

Studies of the dynamics of cold atoms subject to momentum kicks from a standing wave of off-resonant light have played an important role in the development of atom optics and quantum chaos. This system has been used in realizing the quantum  $\delta$ -kicked rotor [1] and the quantum  $\delta$ -kicked accelerator [2, 3]. These experiments have allowed the observation of important time asymptotic behavior such as dynamical localization and quantum resonance [4]. Other interesting classes of kicked systems studied theoretically include the kicked harmonic oscillator [5] and kicked Harper model [6]. In the case of the  $\delta$ -kicked rotor, when the kicking period is close to integer multiples of the primary resonance time (the half-Talbot time), quadratic growth of energy is observed. When a linear potential such as the one produced by the Earth's gravity is added to the kicked rotor Hamiltonian, a kicked accelerator is produced. In this case, when the pulse period is close to a resonance time, symmetry is broken and a fixed momentum per kick can be imparted to the atoms in a specific direction. Atoms which are kicked in such a way are said to be in a Quantum Accelerator Mode (QAM) [3]. Fishman, Guarneri and Rebuzzini (henceforth referred to as FGR) developed a framework called  $\epsilon$ -classical theory [7], where a parameter  $\epsilon$  proportional to the difference of the pulse period from one of the primary resonances plays the role of Planck's constant. In the limit of pulse period very close to a primary resonance time, ( $\epsilon \rightarrow 0$ ), a classical mapping can be used to study the system. In addition to the main QAM discovered in the first experiment [3], the FGR theory predicted the existence of additional QAMs which were subsequently observed by Schlunk *et al.* [8]. More recently, it was found that the QAM can be understood in terms of mode locking and an Arnol'd tongue analysis [9].

Higher order quantum resonances of the  $\delta$ -kicked rotor have been predicted to occur whenever the kicking period is a rational fraction of the primary resonance time [10]. Recently such higher order quantum resonances were observed using a kicked BEC of Na atoms [11] and using a thermal sample of atoms [12]. Therefore it is interesting to ask the question: do such resonances also exist in the quantum  $\delta$ -kicked accelerator and can QAMs be

formed in their vicinity? In this Letter we report the observation of these higher order resonances and their associated QAMs. We also show how  $\epsilon$ -classical theory can be generalized to predict the behavior of QAMs at these resonances. In addition, by treating the standing light wave as a diffraction (phase) grating and using a picture analogous to the fractional Talbot effect in optics [13], we are able to explain the internal momentum state (diffraction order) structure of the QAM.

To begin discussion of QAMs around higher order resonances, we write the Hamiltonian of the quantum  $\delta$ -kicked accelerator in dimensionless units as [7],

$$H = \frac{p^2}{2} - \frac{\eta}{\tau} x + \phi_d \cos(x) \sum_t \delta(t' - t\tau), \quad (1)$$

where  $p = n + \beta$  is momentum in units of two photon recoils ( $\hbar G$ ),  $G = 4\pi/\lambda$ ,  $\lambda$  is the wavelength of the kicking light,  $n$  is the integer part of  $p$ ,  $x$  is the position in units of  $G^{-1}$ ,  $\eta = Mg'T/(\hbar G)$ ,  $g'$  is the acceleration experienced by the atoms during the time between kicks in the direction of the standing wave,  $T$  is the kick period,  $\phi_d$  is the phase modulation depth (and represents the kicking strength), and  $M$  is the atom's mass. The scaled kick period is  $\tau = 2\pi T/T_{1/2}$  where  $T_{1/2} = 2\pi M/(\hbar G^2)$  is referred to as the half Talbot time and  $t'$  is the continuous time variable in units of  $T_{1/2}/2\pi$ . In previous work [2, 3] it was found that the QAM occur whenever  $T$  is close to an integer multiple of  $T_{1/2}$ . It should also be noted that in the frame that is accelerating with  $g'$ , the quasi momentum  $\beta$  (fractional part of the momentum) is conserved. For primary resonances FGR define a parameter  $\epsilon = \tau - 2\pi l = 2\pi(T/T_{1/2} - l)$ , where  $l$  is an integer. The  $\epsilon$  represents the closeness of the kick period to one of the primary resonance times. This enables the one kick evolution operator to be written as  $U_\beta(t) = e^{-\frac{i}{\hbar} \tilde{k} \cos \theta} e^{-\frac{i}{\hbar} \tilde{k} \hat{H}_\beta}$ , where  $\hat{H}_\beta = \frac{1}{2} \frac{\epsilon}{|\epsilon|} \hat{I}^2 + \hat{I} [\pi l' + \tau(\beta + t\eta + \eta/2)]$ ,  $\hat{I} = -i|\epsilon| \frac{d}{d\theta}$ ,  $\tilde{k} = |\epsilon| \phi_d$ ,  $\theta = x \bmod 2\pi$ , and  $l'$  is an integer. By analogy with the kicked rotor, higher order resonances are expected when  $T$  is a rational fraction ( $a/b$ ) of  $T_{1/2}$

[10, 11], where  $a$  and  $b$  are integers. Thus we now define  $\epsilon = \tau - 2\pi a/b$ , which represents the closeness parameter to one of the higher order resonance times. Using this  $\epsilon$  and the condition that at higher order resonances  $(m^2 + 2ml'/a) a/2b$  is an integer, where  $m$  is an integer [10, 11], the evolution operator becomes

$$U_\beta(t) = e^{-\frac{i}{|\epsilon|} \tilde{k} \cos \hat{\theta}} e^{-\frac{i}{|\epsilon|} \hat{H}_\beta^h}, \quad (2)$$

where  $\hat{H}_\beta^h = \frac{1}{2} \frac{\epsilon}{|\epsilon|} \hat{I}^2 + \hat{I} (\pi l'/b + \tau(\beta + t\eta + \eta/2))$ . Since  $|\epsilon|$  plays the role of Planck's constant, in the limit  $\epsilon \rightarrow 0$ , Eq. (2) produces the following classical mapping:

$$\begin{aligned} J_{t+1} &= J_t + \tilde{k} \sin(\theta_{t+1}) + \frac{\epsilon}{|\epsilon|} \eta \tau \\ \theta_{t+1} &= \theta_t + \frac{\epsilon}{|\epsilon|} J_t, \end{aligned} \quad (3)$$

where  $J_t$  is defined as  $J_t = I_t + \frac{\epsilon}{|\epsilon|} (\pi l'/b + \tau(\beta + t\eta + \eta/2))$ . The mapping of Eqs. (3) is very similar to that found for the primary resonances [7]. Hence the  $\epsilon$ -classical theory can be applied to QAM appearing at higher order resonance times. This gives the average momentum transferred to a QAM (in units of  $\hbar G$ ) near  $(a/b) T_{1/2}$  as

$$\bar{p}_{\text{QAM}} = -t \frac{\eta \tau}{\epsilon}. \quad (4)$$

Thus the general signature of a quantum resonance of the quantum  $\delta$ -kicked accelerator is expected to be the asymptotic divergence of a QAM's momentum,  $\bar{p}_{\text{QAM}}$ , to infinity as  $\epsilon \rightarrow 0$ , that is, when the kicking period approaches the resonance time.

To experimentally observe these quantum resonances we subjected a BEC to pulses of standing wave light as described in detail in [2]. Briefly, the BEC was created in an optical trap and consisted of approximately 30000 Rb-87 atoms in the  $F = 1, 5S_{1/2}$  level. After release from the trap, the BEC was kicked by 780 nm light which was 6.8 GHz detuned to the red of the atomic transition. These parameters gave a value for the half-Talbot time of  $T_{1/2} = 33.15 \mu\text{s}$ . This light propagated through two acousto-optic modulators (AOMs) to control the initial momentum of the atoms with respect to the standing wave. This was accomplished by driving the two AOMs with different frequencies. The kicking beam was oriented at  $48^\circ$  to the vertical making  $g' = 6.6 \text{ ms}^{-2}$ . In order to vary the kicking strength  $\phi_d$ , the length of the kicking pulses was adjusted. Typically the pulse length was approximately  $2.5 \mu\text{s}$  giving  $\phi_d \approx 1.5$ . The value of  $\phi_d$  was estimated by comparing the relative population of various diffraction orders after one kick. Note that the population in the  $l$ -th order is given in terms of Bessel functions via  $|J_l(\phi_d)|^2$  [14]. The momentum distribution of the BEC was measured by taking an absorption image  $\approx 8 \text{ ms}$  after the completion of the kicking sequence. Finally it should be noted that the mean field energy was weak enough that it could be ignored, making the Hamiltonian of Eq. (1) a valid approximation.

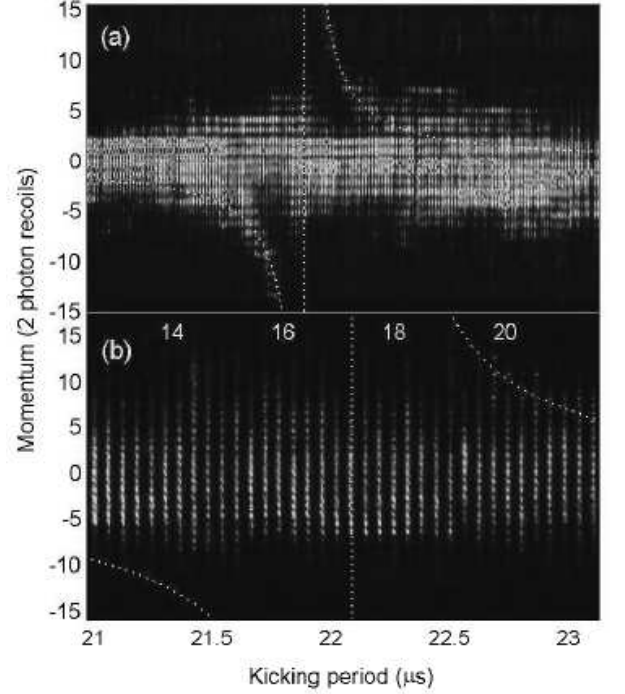


FIG. 1: Horizontally stacked momentum distributions for different kicking periods for (a) 40 kicks across  $(1/2)T_{1/2}$  ( $b = 2$ ) (b) 30 kicks across  $(2/3)T_{1/2}$  ( $b = 3$ ). The initial momentum is chosen such that the part of the mode below the resonance time is populated more strongly in the case of (a) and vice versa in the case of (b). The dotted curve is predicted by the  $\epsilon$ -classical theory of Eq. (4).

Figure 1 shows experimental scans of the kicking period across two different higher order resonances. These figures were generated by horizontally stacking the absorption images each with a different kick period. The dotted curves are the QAM momenta predicted by the  $\epsilon$ -classical theory of Eq. (4). A value for  $\phi_d$  of 1.4 was used in Fig. 1(a) near  $(1/2) T_{1/2}$  and 1.8 in Fig. 1(b) near  $(2/3) T_{1/2}$ . It can be seen that the theory provides a good description of the momentum of the QAM and confirms the presence of the higher order resonances. Both experiments and numerical simulations [15] suggested that higher values of  $\phi_d$  were required to produce observable QAMs as  $b$  was increased. To investigate the properties of higher order resonances of the quantum  $\delta$ -kicked accelerator further, we conducted a series of experiments where kick number was increased at a fixed kicking period close to each of the above resonances. Figure 2(a) is a scan of number of kicks close to  $(2/3)T_{1/2}$ . This hints that the QAM primarily consists of momentum states separated by  $3\hbar G$ . The scan of kick number close to  $(1/2)T_{1/2}$  of Fig. 2(b) shows much more clearly that the QAM is composed of momentum states separated by  $2\hbar G$ . These momentum states are emphasized by arrows in Figs. 2(a) and 2(b). In contrast, at the Talbot time ( $2T_{1/2}$ ), the QAM can include neighboring momentum states as seen

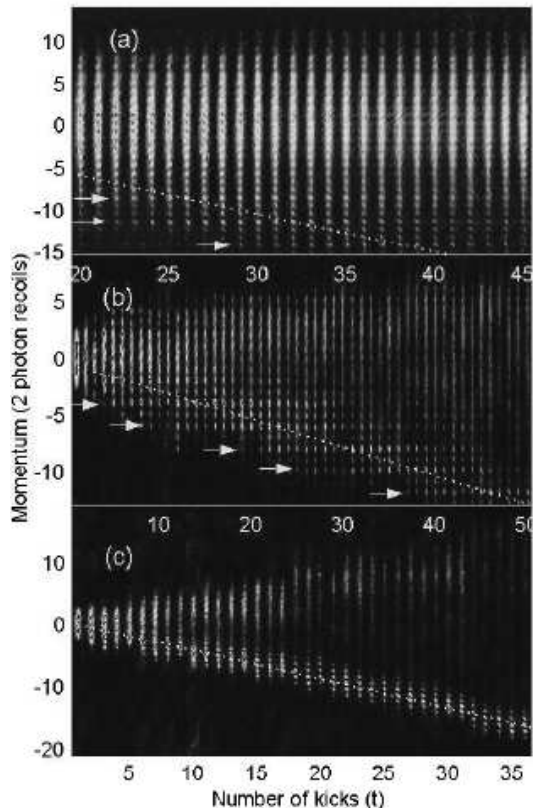


FIG. 2: Horizontally stacked momentum distributions as a function of number of kicks ( $t$ ) for (a)  $T = 22.68 \mu\text{s}$ , which is close to  $(2/3)T_{1/2}$ , (b)  $T = 17.1 \mu\text{s}$ , which is close to  $(1/2)T_{1/2}$ , and (c)  $T = 72.4 \mu\text{s}$  which is close to  $2T_{1/2}$ . Note the different axes for (a), (b) and (c). The arrows in (a) and (b) show that primarily orders separated by  $b\hbar G$  participate in each of the QAMs. Dotted lines show the fit to  $\epsilon$ -classical theory of Eq. (4).

from Fig. 2(c). This behavior suggests that the QAM around the higher order resonances can form whenever the momentum orders separated by  $b\hbar G$  rephase during the time between the kicks. This is analogous to what has been postulated to occur (but never directly observed) for the kicked rotor resonances [11], and is consistent with what is known of the fractional Talbot effect [13].

Given this structure for the QAMs it is now possible to explain why an increased  $\phi_d$  is needed to observe QAMs near resonances with larger  $b$ . This is due to the fact that a kick diffracts atoms into a wider range of momentum orders for larger  $\phi_d$  so that the orders separated by  $b\hbar G$  can be more readily populated. Recall that the population of a momentum state  $l\hbar G$  is proportional to  $|J_l(\phi_d)|^2$ . This in turn explains why it is progressively more difficult to observe the QAMs and associated resonances as  $b$  becomes greater. Since  $\phi_d$  must be increased in this situation, the distribution of momentum states that do not participate in a QAM broadens. This can mask the presence of a QAM in either a scan of kick period or kick number especially in the case of experiments with only a

few kicks.

Although the  $\epsilon$ -classical model can not explain the momentum state structure of a QAM as seen in Figs. 1 and 2, it predicts accurately the average momentum imparted to a QAM. FGR attribute the existence of a QAM to the presence of stable islands in the phase space map generated by Eq. (3), here shown in Fig. 3. The momentum axes in these maps are over a range of  $1\hbar G$ . The appearance of two islands in Fig. 3(b) shows that the repetition in initial momentum ( $p_i$ ) for a QAM to appear at  $(2/3)T_{1/2}$  is  $0.5 \hbar G$ . For the case of the kicked rotor, at a resonance with a given  $a$  and  $b$  a quadratic growth of mean energy occurs when the initial momentum is a multiple of  $\hbar G/a$  [7]. To calculate a similar result for the  $\delta$ -kicked accelerator we can apply a method analogous to that used to first understand the QAM at the primary quantum resonances [14]. This is done by setting the difference in phase acquired by momentum states separated by  $b\hbar G$  during the free evolution between any two kicks to an integer multiple of  $2\pi$ . This gives the condition on initial momentum (in units of  $\hbar G$ ) at which a QAM appears as

$$p_i = \frac{2\pi j}{\tau b} + \frac{b}{2} - \frac{\eta}{2}, \quad (5)$$

where  $j$  is an integer. The last term here is small enough to be ignored for the parameters used in our experiment. It can be seen that the QAM in the  $\delta$ -kicked accelerator are spaced by  $\Delta p_i = 1/\tau b \approx 1/a$ . To observe the

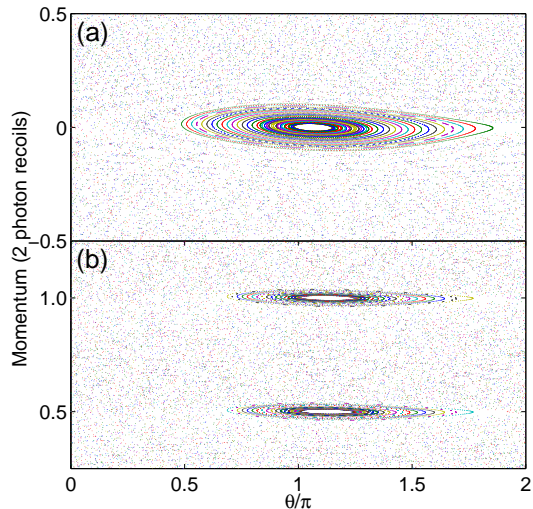


FIG. 3: Phase space plot of the map from (3) for (a)  $\phi_d = 1.5$  and  $T = 17.1 \mu\text{s}$ , which is close to  $(1/2)T_{1/2}$ , and (b)  $\phi_d = 1.8$  and  $T = 22.6 \mu\text{s}$  which is close to  $(2/3)T_{1/2}$ . The stable islands represent QAMs.

detailed phase space structure experimentally, it is necessary that the momentum width should be much narrower than  $\hbar G$ . In our experiments the BEC had a momentum width of  $0.1 \hbar G$  which makes it an excellent candidate for this task. Figure 4(a) shows the results from experiments in which the effective initial momentum of the

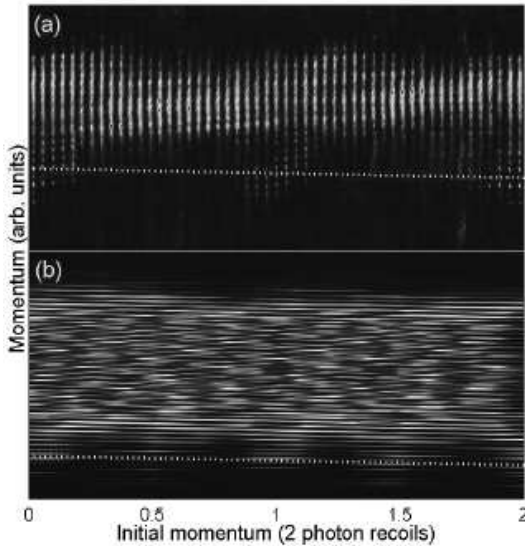


FIG. 4: Horizontally stacked momentum distributions as a function on the initial momentum of the BEC before kicking with (a) 30 kicks with a period of  $17.1 \mu\text{s}$  (b) a numerical simulation for 70 kicks with the period of  $22.6 \mu\text{s}$ . The position of the modes are indicated by the dotted lines

BEC was changed by moving the standing wave using a difference in frequency between the kicking AOMs. The kicking period was near  $(1/2)T_{1/2}$  in Figs. 4(a). The numerical simulation for  $(2/3)T_{1/2}$  is displayed in Fig. 4(b). Experiments in which the initial momentum was scanned for  $(2/3)T_{1/2}$  showed no distinct features. This was most likely because the modes at this resonance are very weak,

as can be seen from Figs. 1(b) and 2(a). The numerical simulations show observable modes emerging after 70 kicks. Our experiments can not reach 70 kicks with the large value of  $\phi_d$  necessary to observe the mode at  $(2/3)T_{1/2}$ . Nevertheless, the simulations do show that for  $(2/3)T_{1/2}$  the mode appears four times in a range of  $2\hbar G$ . Thus, the initial momenta at which the modes can exist are separated by  $\hbar G$  in the case of  $(1/2)T_{1/2}$  ( $a = 1$ ) and  $\hbar G/2$  for  $(2/3)T_{1/2}$  ( $a = 2$ ) in agreement with Eq. (5). The periodicity of QAMs was deduced from the separation of QAMs in Fig. 4.

In conclusion, we have experimentally demonstrated the existence of higher order resonances in the quantum  $\delta$ -kicked accelerator. This was possible through the observation of QAMs near these resonances. The  $\epsilon$ -classical theory of FGR was generalized to predict the behavior of the system near the higher order resonances. Furthermore, we were able to explore the phase space structures produced by maps of the generalized theory. The momentum transferred to a QAM at a resonance was in agreement with the theory. The narrow momentum distribution of the BEC allowed us to observe the momentum state structure of the QAMs. It was found that QAMs near higher order resonances can have a very different structure which is reminiscent of the momentum state structure produced by the fractional Talbot effect. This work opens the door towards the study of higher order QAMs near higher order resonances. Other interesting questions include the effect of stronger mean field interactions and the enhancement of QAMs using more complex initial states.

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